Inversely, to find $(D)^m$ in terms of $D^m$, $D^{m-1}$, .. $D$, we have $m$ equations linear in $(D)^m$, $(D)^{m-1}$, .. $(D)$ (obtained by assuming for $m$ in (1) the values 1, 2, 3, .. $m$) from which, if $(m-k)_i$ = the sum of the products of the natural numbers from 1 to $m-k$ inclusive taken $i$ at a time,

$$(D)^m = \begin{vmatrix} D^m & -(m-1)_1 & (m-1)_2 & \dots & \mp(m-1)_{m-1} \\ D^{m-1} & 1 & (m-2)_1 & \dots & \pm(m-2)_{m-2} \\ D^{m-2} & 0 & 1 & \dots & \mp(m-3)_{m-3} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ D & 0 & 0 & \dots & 1 \end{vmatrix}$$

By another familiar process applied to the symbols $D^m$ $[=(D)^{m\prime}]$ and $(D)^m$ we get also

$$(D)^m = \Sigma \frac{\Delta^i 0^m}{i!} D^i.$$

*Examples of* (1).—1. To change the independent variables in $D^m$ from $\alpha_1, \alpha_2, ..$ to $\theta_1, \theta_2, ..$, where $\alpha_1 = \varepsilon^{\theta_1}$, $\alpha_2 = \varepsilon^{\theta_2}, ..$; so that

$$\alpha_1 \frac{d}{d\alpha_1} = \frac{d}{d\theta_1}, \; \alpha_2 \frac{d}{d\alpha_2} = \frac{d}{d\theta_2}, \dots \qquad \left(\left(\alpha_1 \frac{d}{d\alpha_1} + \alpha_2 \frac{d}{d\alpha_2} + ..\right)\right)^{m\prime}$$

becomes by the transformation

$$\left(\frac{d}{d\theta_1} + \frac{d}{d\theta_2} + ..\right)^{m\prime}. \quad \therefore \text{ by (1) } D^m = \left(\frac{d}{d\theta_1} + \frac{d}{d\theta_2} + ..\right)^{m\prime}.$$

For a special case see Todhunter's Dif. Cal. Art. 208.

2. By Euler's theorem concerning a homogeneous function $\varphi(\alpha_1, \alpha_2, ..)$, of $n$ dimensions, $F((D))\varphi = F(n)\varphi$; $\therefore$ $(D)^{m\prime} \varphi = n^{m\prime}\varphi$; $\therefore$ by (1)

$$D^m \varphi = n^{m\prime} \varphi.$$

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## *SOLUTION OF A PROBLEM.*

BY PROF. E. W. HYDE, UNIVERSITY OF CINCINNATI.

*Problem* —To show that $\cos^p\varphi \sin^q\varphi$ can be expanded into a series of *cosines* of multiples of $\varphi$ when $q$ is *even*, and into a series of *sines* of multiples of $\varphi$ when $q$ is *odd*.

First, suppose $q$ even and $= 2n$, say. Then

$$\cos^p\varphi \sin^{2n}\varphi = \cos^p\varphi(\sin^2\varphi)^n = \cos^p\varphi(1-\cos^2\varphi)^n$$

$$= \cos^p\varphi - n\cos^{p+2}\varphi + \frac{n(n-1)}{2!}\cos^{p+4}\varphi - \&c.$$

Each term of this series can be expanded into a series of cosines of multiples of $\varphi$.

Second, suppose $q$ odd and $= 2n + 1$ say, and also let $p$ be even and $=2m$. Then

$$\cos^{2m}\varphi \sin^{2n+1}\varphi = (\cos^2\varphi)^m \sin^{2n+1}\varphi = \sin^{2n+1}\varphi(1-\sin^2\varphi)^m$$
$$= \sin^{2n+1}\varphi - m \sin^{2n+3}\varphi + \frac{m(m-1)}{2!} \sin^{2n+5}\varphi - \&c.$$

Since each term of the right hand member is an odd power of $\sin\varphi$, each term may be expanded into a series of sines of multiples of $\varphi$.

Finally let $q = 2n+1$, and $p = 2m+1$ so that both are odd. Then, if we suppose $m < n$,

$$\begin{aligned}\cos^{2m+1}\varphi \sin^{2n+1}\varphi &= (\sin\varphi\cos\varphi)^{2m+1}(\sin^2\varphi)^{n-m}\\ &= (\tfrac{1}{2})^{m+n+1}(\sin 2\varphi)^{2m+1}(1-\cos 2\varphi)^{n-m};\end{aligned}$$

while if we suppose $m > n$ we have

$$\begin{aligned}\cos^{2m+1}\varphi \sin^{2n+1}\varphi &= (\sin\varphi\cos\varphi)^{2n+1}(\cos^2\varphi)^{m-n}\\ &= (\tfrac{1}{2})^{m+n+1}(\sin 2\varphi)^{2n+1}(1+\cos 2\varphi)^{m-n}.\end{aligned}$$

From the similarity of these two forms it evidently makes no difference whether $m$ be greater or less than $n$, we will use then the first expression, i. e., regard $m$ as less than $n$.

$$\therefore \cos^{2m+1}\varphi \sin^{2n+1}\varphi = (\tfrac{1}{2})^{m+n+1}(\sin 2\varphi)^{2m+1}$$
$$\times\left[1-(n-m)\cos 2\varphi + \frac{(n-m)(n-m-1)}{2!}\cos^2 2\varphi - \&c.\right].$$

Now the product of $(\sin 2\varphi)^{2m+1}$ into any even power of $\cos 2\varphi$ comes under the second case, and we have therefore only to consider the products containing *odd* powers.

The exponent of the highest odd power will be either $n-m$ or $n-m-1$; call it $2m' + 1$: then treating that product as before we have, supposing $m' < m$,

$$(\cos 2\varphi)^{2m'+1}(\sin 2\varphi)^{2m+1} = (\tfrac{1}{2})^{m+m'+1}(\sin 4\varphi)^{2m'+1}[1-\cos 4\varphi]^{m-m'}.$$

Repeating this process successively we may finally obtain either

$$\sin[2^r\varphi]\,[1\mp\cos(2^r\varphi)] = \sin(2^r\varphi)\mp\tfrac{1}{2}\sin(2^{r+1}\varphi),$$

or

$$\sin(2^s\varphi)\cos(2^s\varphi) = \tfrac{1}{2}\sin(2^{s+1}\varphi).$$

To show that this will be the result we will find the degrees of the successive left hand members. That of the first is

$$2m+1+2n+1 = 2(m+n+1).$$

That of the second, supposing $n-m$ to be odd, is

$$2m+1+n-m = m+n+1.$$

That of the third, supposing $m-m'$ odd, is

$$n-m+m-\frac{n-m-1}{2} = \frac{n+m+1}{2}.$$

Thus the degree each time is half what it was before. If $n-m$, $m-m'$, $m'-m''$, etc., be *even*, we shall have for the successive degees,

$$\begin{aligned}
&\text{1st, } 2m+1+2n+1=2(m+n+1),\\
&\text{2nd, } 2m+1+2m'+1=2(m'+m+1)=m+n,\\
&\text{3rd, } 2m'+1+2m''+1=2(m''+m'+1)=\tfrac{1}{2}(m+n-2),\\
&\text{4th, } 2m''+1+2m'''+1=2(m'''+m''+1)=\tfrac{1}{4}(m+n-2), \text{ etc.}
\end{aligned}$$

After the second, the degree is each time half the preceding. If the exponents $n-m$, $m-m'$, etc., were some even and some odd the result would be between these two.

A numerical example is added.—Let $m=9$ and $n=14$. Then

$$\begin{aligned}
\cos^{19}\varphi\sin^{29}\varphi &= (\tfrac{1}{2})^{24}(\sin 2\varphi)^{19}[1-\cos 2\varphi]^5,\\
(\cos 2\varphi)^5(\sin 2\varphi)^{19} &= (\tfrac{1}{2})^{12}(\sin 4\varphi)^5[1-\cos 4\varphi]^7,\\
(\cos 4\varphi)^7(\sin 4\varphi)^5 &= (\tfrac{1}{2})^6(\sin 8\varphi)^5[1+\cos 8\varphi],\\
\cos 8\varphi\,(\sin 8\varphi)^5 &= (\tfrac{1}{2})^3(\sin 16\varphi)[1-\cos 16\varphi]^2,\\
\cos 16\varphi\sin 16\varphi &= \tfrac{1}{2}\sin 32\varphi.
\end{aligned}$$

If in the expansion of the right-hand member of the first equation we take $(\cos 2\varphi)^3$ instead of $(\cos 2\varphi)^5$ we have

$$\begin{aligned}
(\cos 2\varphi)^3(\sin 2\varphi)^{19} &= (\tfrac{1}{2})^{11}(\sin 4\varphi)^3[1-\cos 4\varphi]^8,\\
(\cos 4\varphi)^7(\sin 4\varphi)^3 &= (\tfrac{1}{2})^5(\sin 8\varphi)^3[1+\cos 8\varphi]^2,\\
\cos 8\varphi\,(\sin 8\varphi)^3 &= (\tfrac{1}{2})^2(\sin 16\varphi)[1-\cos 16\varphi],\\
&= (\tfrac{1}{2})^2[\sin 16\varphi-\tfrac{1}{2}\sin 32\varphi].
\end{aligned}$$

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## *DISCUSSION OF AN EQUATION.*

BY JOHN BORDEN, CHICAGO, ILL.

As equations of the fifth or less degree are reducible to the form

$$y^n+py+q=0, \tag{1}$$

$n$ being an integer and positive, its discussion is of interest.

QUESTION.—Determine the number of real roots, their limits, and sign, for all real values of $p$ and $q$.

ANSWER:—I. *If* n *is odd and* p *is positive,* there is only one real root, whose sign is contrary to that of $q$ and whose limits are $q\div p\times-0$, and $q\div p\times-1$.

II. *If* n *is odd and* p *is negative,* there are three real roots when

$$\frac{p^n}{q^{n-1}} \begin{matrix}=\\>\end{matrix} \frac{n^n}{(n-1)^{n-1}},$$

two of which have the same sign as $q$, and become equal to each other, and